Plateau's problem for infinite contours

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Dedicated to Erhard Heinz on the occasion of his 85^{th} birthday

1 Introduction and Result.

Already B. Riemann described a general procedure how to construct simply connected minimal surfaces bounded by straight line segments, finite or infinite [13]. As special cases he considered the following boundary configurations: (i) two half lines with a common endpoint together with a full line parallel to the plane of the two half lines, (ii) three pairwise skew lines. Later E. Neovius observed that the above configuration (i) may also be spanned by doubly connected minimal surfaces and he extended Riemann's method correspondingly [11]. These authors gave the solution in terms of integrals depending on certain parameters which had to be chosen in order to satisfy the given boundary condition. The question of which boundary configurations can actually be realized in this way was left open, apart from some special cases. In recent years the problem of minimal surfaces spanning unbounded piecewise linear contours gained new interest leading to existence, uniqueness, and multiplicity results for some special families of such contours [4,8,9].

Analogous to the classical Plateau problem [2, Chpt. 4.3] it seems natural to pose the problem of the existence of minimal surfaces spanning *general* unbounded curves. In this paper we solve this problem in the case of simply connected surfaces and connected boundaries from a suitably restricted class. This class contains all properly embedded curves consisting of finitely many polynomial pieces. The case of minimal graphs with boundary data $\pm\infty$ was investigated in [7]. The leading idea in our approach is to work in

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the class of minimal surfaces with quadratic area growth; that is to say that the area of such minimal surfaces inside a ball of radius r grows no faster than const r^2. Our conditions on the admissible contours are tailored correspondingly. Therefore, if the boundary is a straight line, the methods of this paper produce a half plane as solution. The half helicoid, having the same boundary, is out of reach of our technique.

We use the following notations:

$$\begin{aligned} B_r(p) &:= \{x \in \mathbb{R}^3 \,|\, |x-p| < r\}, \; r > 0, \, p \in \mathbb{R}^3, \\ B_r &:= B_r(0), \\ D_r(w) &:= \{z \in \mathbb{R}^2 \,|\, |z-w| < r\}, \, r > 0, \, w \in \mathbb{R}^2 \\ D_r &:= D_r(0), \\ H &:= \{z = (u,v) \in \mathbb{R}^2 \,|\, v > 0\}\,. \end{aligned}$$

We now list our conditions on the admissible contours. Let Γ be a noncompact, properly embedded curve in $\mathbb{R}^3$, piecewise of class $C^{1,\alpha}$ for some $\alpha > 0$. We assume that $0 \in \Gamma$ and for $R > 0$ we denote by Γ_R the connected component of $\Gamma \cap B_R$ containing 0. In slight abuse of notation we also use the symbol Γ for the arc-length representation $\Gamma : \mathbb{R} \to \mathbb{R}^3$, normalized by the condition $\Gamma(0) = 0$. We require furthermore:

(1.1) there is $\delta > 0$ such that $|p - q| \geq \delta$ for all points p, q in different components of $\Gamma \backslash \Gamma_1$,

(1.2) setting $\gamma(s) = |\Gamma(s)|^{-1}\Gamma(s)$ there holds

(i) $|\langle \gamma(s),\ \Gamma'(s)\rangle| \to 1 \ (|s| \to +\infty)$

and

(ii) $\int\limits_{\Gamma\backslash\Gamma_1} |\Gamma(s)|^{-1}\sqrt{1 - \langle \gamma(s),\ \Gamma'(s)\rangle^2}\, ds < +\infty\,.$

Roughly speaking, the condition (1.2) requires the curve Γ to tend to infinity in a sufficiently straight manner. Actually, (1.2) does imply that the tangent vector has limits at the ends of Γ, but (1.2) does not imply that the ends stay within bounded distance to some straight lines; for example, all polynominal ends satisfy (1.2). The condition (1.2) (ii) expresses the fact that the cone over Γ has quadratic area growth.

We shall prove the following

Theorem 1.1 *Let Γ be a properly embedded curve in $\mathbb{R}^3$, piecewise of class $C^{1,\alpha}$ and let (1.1)and (1.2) be satisfied. Then there exists a proper map $X \in C^0(\bar{H}, \mathbb{R}^3) \cap C^\infty(H, \mathbb{R}^3)$ which is a harmonic and conformal immersion on H and $X|\partial H$ parametrizes Γ in a strictly monotonic way.*

Remark 1.2 *The results of Meeks-Yau [10] imply that X is an embedding provided that Γ is contained in the boundary of a mean-convex set.*

Remark 1.3 *Our proof gives the corresponding result for the ambient space $\mathbb{R}^n$, $n \geq 4$, with the exception that X may have isolated branch point singularities.*

Remark 1.4 *If Γ has finite total curvature, so does X.*

In the last section of the paper we shall investigate the asymptotical shape of the surfaces obtained in Theorem 1.1.

2 The proof.

Lemma 2.1 *There are numbers $\rho_0 > 0$, $C > 0$ with the following properties:*

(i) $|\Gamma(s)|$ *is strictly increasing for* $s \geq \rho_0$ *and strictly decreasing for* $s \leq -\rho_0$,

(ii) $\lim\limits_{s \to \pm\infty} |\Gamma(s)|/|s| = 1$ *and* $|\Gamma(s)| \geq C^{-1}|s|$ *for* $|s| \geq 1$,

(iii) *length* $(\Gamma \cap B_R) \leq 2C\,R$ *and length* $(\Gamma \cap B_R)/R \to 2$ $(R \to +\infty)$,

(iv) $\Gamma \cap B_R$ *is connected for* $R \geq \rho_0$,

(v) the following limits exist:

$$\gamma^+ := \lim_{s \to +\infty} \gamma(s), \;\; \gamma^- := \lim_{s \to -\infty} \gamma(s).$$

Proof. We have

$$\frac{d}{ds}|\Gamma(s)| = \langle \gamma(s), \Gamma'(s) \rangle$$

and it follows therefore from (1.2) that $\langle \gamma(s), \Gamma'(s) \rangle \to 1 (s \to +\infty)$, $\langle \gamma(s), \Gamma'(s) \rangle \to -1 (s \to -\infty)$, proving (i). The rule of de l'Hôpital implies $|\Gamma(s)|/|s| \to$

$1 (s \to \pm\infty)$ and since $\Gamma(0) = 0$ and Γ is embedded the existence of a constant C such that $|\Gamma(s)| \geq C^{-1}|s|$ follows. The first inequality in (iii) is a direct consequence of (ii). With $\Gamma \cap B_R = \Gamma([s_R^-, s_R^+])$ we have R^{-1} length $(\Gamma \cap B_R) = R^{-1}(s_R^+ - s_R^-) = s_R^+/|\Gamma(s_R^+)| - s_R^-/|\Gamma(s_R^-)| = 2$ by (ii). Assertion (iv) follows immediately from (i). To prove (v) compute

$$\frac{d}{ds}\gamma(s) = |\Gamma(s)|^{-1}\left(\Gamma'(s) - \langle \gamma(s), \Gamma'(s)\rangle \gamma(s)\right)$$

and

$$|\frac{d}{ds}\gamma(s)| = |\Gamma(s)|^{-1}\sqrt{1 - \langle \gamma(s), \Gamma'(s)\rangle^2}\,.$$

Therefore, by assumption (1.2) γ' is integrable and (v) follows. □

The existence of a solution to the Plateau problem with the infinite boundary curve Γ as described in Theorem 1.1 will be demonstrated by approximation with a sequence of compact minimal surfaces which are obtained as solutions to the classical Plateau problem with boundary curves $\Gamma_R \cup \beta_R$, $R \geq 1$, where β_R is part of a great circle on the sphere ∂B_R joining the endpoints $\Gamma(s^-(R))$, $\Gamma(s^+(R))$ of Γ_R, $s^-(R) < 0 < s^+(R)$. Clearly, if $R \to +\infty$, then Γ_R converges to Γ and β_R disappears at infinity. The Douglas-Radò existence theorem [2] guarantees the existence of disc type minimal surfaces spanning $\Gamma_R \cup \beta_R$; more precisely, there exist mappings $X^R \in C^0(\bar{H}\,|\mathbb{R}^3) \cap C^\infty(H, \mathbb{R}^3)$ with the following properties

(2.1) X^R is harmonic and conformal on H,

(2.2) X^R maps $\partial H \cup \infty$ topologically onto $\Gamma_R \cup \beta_R$,

(2.3) X^R minimizes simultaneously area and Dirichlet energy among all maps $Y \in C^0(\bar{H} \cup \infty, \mathbb{R}^3)$ which fulfil the boundary condition (2.2).

As a normalization condition we may furthermore require that

(2.4) $X^R(-1) = \Gamma(s^-(1))$, $X^R(0) = \Gamma(0) = 0, X^R(1) = \Gamma(s^+(1))\,.$

An important remark must be made on the geometric regularity of the surfaces X^R: whereas the classical Douglas-Radò theorem leaves the possibility of branch-point singularities open, it was much later shown through the work of Ossermann [12], Gulliver [6], and Alt [1], that minimizing surfaces like the X^R are actually immersed in the interior.

In the following we denote by $A(M)$ the area of a surface M and by

$$E(X,\Omega) := \frac{1}{2}\int\limits_{\Omega} \big(|X_u|^2 + |X_v|\big)\, dudv$$

the Dirichlet energy of a mapping $X : \Omega \to \mathbb{R}^3$.

We start our computation with an estimate of the area of X^R which we obtain by comparison with the cone C^R over $\Gamma_R \cup \beta_R$ with vertex at the origin, assuming that β_R is a shortest connection of the endpoints of Γ_R on ∂B_R. With

$$\alpha(R) := \sphericalangle(\gamma(s^-(R)),\, \gamma(s^+(R)) \in (0,\pi]$$

we obtain

$$A(C^R) = \left(\frac{1}{2}\,\alpha(R) + \varepsilon(R)\right) R^2, \tag{2.5}$$

where

$$\begin{aligned}
\varepsilon(R) &= \frac{1}{2}R^{-2}\int\limits_{s^-(R)}^{s^+(R)} |\Gamma(s)|\sqrt{1-\langle\gamma(s),\ \Gamma'(s)\rangle^2}\, ds \\
&\leq \frac{1}{2R}\int\limits_{s^-(R)}^{s^+(R)} \frac{s}{R}\sqrt{1-\langle\gamma(s),\, \Gamma'(s)\rangle^2}\, ds \\
&\to 0 (R \to +\infty)
\end{aligned}$$

since $|s^{\pm}(R)| \leq CR$ by Lemma 2.1 (ii) and $|\langle\gamma(s),\, \Gamma(s)\rangle| \to 1\,(|s| \to +\infty)$ by hypothesis (1.2) (i). Consequently we have

$$A(X^R) \leq \left(\frac{1}{2}\alpha(R) + \varepsilon(R)\right) R^2 \tag{2.6}$$

where $\alpha(R) \to \alpha := \sphericalangle(\gamma^-,\gamma^+)$ and $\varepsilon(R) \to 0$ for $R \to +\infty$ (cf. Lemma 2.1 (v)).

Next we employ the monotonicity formula to derive local (in space) area bounds from (2.6) which are uniform with respect to R.

Lemma 2.2 *The surface $M^R := X^R(H)$ fulfills the estimate*

$$
(2.7)\qquad \begin{aligned}
\tfrac{1}{r^2}\, A(M^R \cap B_r) \;&\le\; \tfrac{1}{2}\,\alpha(R) + \varepsilon(R) \\
&\quad + \int\limits_{\Gamma\cap B_R} \frac{\sqrt{1-\langle \gamma(s),\, \Gamma'(s)\rangle^2}}{\max(r, |\Gamma(s)|)}\, ds,\ \ 0 < r \le R\,.
\end{aligned}
$$

Proof. The monotonicity formula [3,15] yields

$$
(2.8)\qquad \frac{\partial}{\partial\rho}\left(\frac{1}{\rho^2}\, A(M^R \cap B_\rho)\right) \ge -\frac{1}{\rho^3} \int\limits_{\Gamma\cap B_\rho} \langle x, \nu\rangle ds
$$

for $0 < \rho < R$, where x is the position vector and ν the outward pointing unit normal vector in the surface M^R. Since we assume Γ to be piecewise of class $C^{1,\alpha}$, $\alpha > 0$, the boundary regularity theorem for minimal surfaces [2, Chpt. 7.3] justifies the applicability of the divergence theorem, yielding (2.8). With $V(x) := |x|^{-1}x$ we obtain

$$
\begin{aligned}
\Big|\int\limits_{\Gamma\cap B_\rho} \langle x, \nu\rangle\, ds\Big| \;&=\; \Big|\int\limits_{\Gamma\cap B_\rho} |x|\langle V(x) - \langle V(x), \Gamma'\rangle\, \Gamma', \nu\rangle\, ds\Big| \\
&\le\; \rho \int\limits_{\Gamma\cap B_\rho} |V - \langle V, \Gamma'\rangle \Gamma'| ds = \rho \int\limits_{\Gamma\cap B_\rho} \sqrt{1-\langle V, \Gamma'\rangle^2}\, ds\ .
\end{aligned}
$$

Combining this with (2.8) we get upon integration

$$
\begin{aligned}
\frac{1}{R^2} A(M^R) - \frac{1}{r^2} A(M^R \cap B_r) \;&\ge\; -\int_r^R \frac{1}{\rho^2} \int\limits_{\Gamma\cap B_\rho} \sqrt{1-\langle V, \Gamma'\rangle^2}\, ds\, d\rho \\
&=\; -\int\limits_{\Gamma\cap B_R} \sqrt{1-\langle V, \Gamma'\rangle^2} \int\limits_{\max(r,|x|)}^{R} \frac{d\rho}{\rho^2}\, ds \\
&\ge\; -\int\limits_{\Gamma\cap B_R} \frac{\sqrt{1-\langle V, \Gamma'\rangle^2}}{\max(r, |x|)}\, ds\,.
\end{aligned}
$$

Taking (2.6) into account we obtain the statement of the lemma. □

An immediate consequence of Lemma 2.2 and our hypothesis (1.2) (ii) is

Corollary 2.3 *There is a constant a, only depending on Γ, such that*

$$A(M^R \cap B_r) \le ar^2, \ 1 \le r \le R.$$

Like in [16] the local area estimates in space can be converted into local energy estimates for the parametrization X^R. For this purpose we define

$$\begin{aligned} \Omega(Y,\rho) &:= (Y)^{-1}(B_\rho), \\ C(Y,\rho) &:= \text{the connected component of } \Omega(Y,\rho) \text{ containing } [-1,1], \end{aligned}$$

whenever $Y : \bar{H} \to \mathbb{R}^3$ is a continuous map. We remark that $[-1,1] \subset \Omega(X^R,\rho)$ for $\rho \in [1,R]$ since $X^R([-1,1]) = \Gamma_1 \subset B_1$ because of (2.4).

By a rescaling we may assume that the number ρ_0 in Lemma 2.1 is 1, i.e. $s \mapsto |\Gamma(s)|$ is monotone for $|s| \ge 1$.

Lemma 2.4

(i) $dist(\partial C(X^R,\rho)\backslash\mathbb{R},\ \partial C(X^R,2\rho)\backslash\mathbb{R}) \ge exp(-16\pi a),\ 1 \le \rho \le R/2,$

(ii) $dist(0, \partial C(X^R,\rho)\backslash\mathbb{R}) \ge c\ln\rho,\ 2 \le \rho \le R.$

Here a is the constant from Lemma 2.3, ∂H is identified with $\mathbb{R}$, and $c > 0$ is some further constant.

Proof.

(i) We choose points $z \in \overline{\partial C(X^R,\rho)\backslash\mathbb{R}}$ and $w \in \overline{\partial C(X^R,2\rho)\backslash\mathbb{R}}$ such that $|z-w| = \text{dist}(\partial C(X^R,\rho)\backslash\mathbb{R},\ \partial C(X^R,2\rho)\backslash\mathbb{R}) =: \delta_0$. We may assume that $\delta_0 < 1$. Let $r \in (\delta_0, 1)$.

Claim I: $\partial D_r(z) \cap C(X^R,\rho) \neq \emptyset$. This is obvious if $\partial D_r(z)$ meets $[-1,1]$. If this is not the case then $[-1,1]$ lies in the exterior of $D_r(z)$ since $[-1,1] \subset D_r(z)$ is impossible because of $r < 1$. If then $\partial D_r(z) \cap C(X^R,\rho)$ were empty, it would follow that $\partial D_r(z)$ separates $z \in C(X^R,\rho)$ from $[-1,1]$, contradicting the definition of $C(X^R,\rho)$.

Claim II: There exists $\zeta \in \partial D_r(z)$ such that $|X^R(\zeta)| \ge 2\rho$. Since $w \in D_r(z)$ and $|X^R(w)| = 2\rho$, it follows from the maximum principle that there must be a point $\zeta_0 \in \partial(D_r(z) \cap H)$ such that $|X^R(\zeta_0)| \geq 2\rho$. In case that $\zeta_0 \in \partial H$ we may use the monotonicity of $s \mapsto |\Gamma(s)|$ for $|s| \geq 1$ and the assumption $\rho \geq 1$ to infer the existence of $\zeta_1 \in \partial D_r(z) \cap \partial H$ such that $|X^R(\zeta_1)| \geq 2\rho$.

Claim I and II being proved we infer the existence of a subarc σ_r of $\partial D_r(z)$ which connects $C(X^R, \rho)$ with $\overline{\partial C(X^R, 2\rho)\backslash\mathbb{R}}$ within $C(X^R, 2\rho)$.

According to the Lemma of Courant-Lebesgue [2, Chpt. 4.3] we may choose $r \in (\delta_0, \sqrt{\delta_0})$ in such a way that the following inequalities hold:

$$\rho^2 \leq \left(\int\limits_{\sigma_r} \left| \frac{\partial X^R}{\partial \theta} \right| d\theta \right)^2 \leq \frac{4\pi}{-\ln\, \delta_0}\, E(X^R, \Omega(X^R, 2\rho)) \leq \frac{16\pi a}{-\ln\, \delta_0}\, \rho^2 \,,$$

using Corollary 2.3 in the last step. This is statement (i).

(ii) We start with $\rho = 2$. Assuming that $\delta_0 := \operatorname{dist}(0, \partial C(x^R, 2)\backslash\mathbb{R}) < 1$ we again find by means of the Courant-Lebesgue lemma a radius $r \in (\delta_0, \sqrt{\delta_0}\,)$ such that a subarc σ_r of $\partial D_r(0) \cap C(X^R, 2)$ joins $[-1, 1]$ with $\partial C(X^R, 2)\backslash\mathbb{R}$. Since $X^R([-1, 1]) \subset B_1$ we conclude

$$1 \leq \left(\int\limits_{\sigma_r} \left| \frac{\partial X^R}{\partial \theta} \right| d\theta \right)^2 \leq \frac{4\pi}{-\ln\, \delta_0}\, E(X^R, C(X^R, 2)) \leq \frac{16\pi a}{-\ln\, \delta_0} \,,$$

and hence

$$\operatorname{dist}(0, \partial C(X^R, 2)\backslash\mathbb{R}) \geq \exp(-16\pi a)\,.$$

With the help of (i) we then conclude inductively

$$\operatorname{dist}(0, \partial C(X^R, 2^{k+1})\backslash\mathbb{R}) \geq k \exp(-16\pi a)$$

for all $k \in \mathbb{N}$ with $2^{k+1} \leq R$. From this the statement (ii) follows. □

Combining Corollary 2.3 and Lemma 2.4 we obtain

Corollary 2.5

$|X^R| \leq e^{r/c}$ on $D_r(0) \cap \bar{H}$ and $E(X^R, D_r(0) \cap H) \leq a\, e^{2r/c}$ for $r \in [c \ln 2, c \ln R]$.

We are now ready to pass to the limit $R \to +\infty$.

Proposition 2.6

(i) There are sequences $R_k \to +\infty (k \to \infty)$ *such that the maps* X^{R_k} *converge locally uniformly on* $\bar{H}$ *to a map* $X \in C^0(\bar{H}, \mathbb{R}^3) \cap C^\infty(H, \mathbb{R}^3)$ *for* $k \to \infty$. *The map* X *is harmonic on* H *and* X *maps* ∂H *strictly monotonically into* Γ.

(ii) Any such limit map is a conformal immersion of H *into* $\mathbb{R}^3$.

Proof.

(i) Let $r > 0$ be fixed. Since $X^R(\partial H) = \Gamma_R \cup \beta_R$, it follows from Corollary 2.4 that $X^R(\partial H \cap D_r(0)) \subset \Gamma$ for $R > e^{r/c}$. Then, using a well known argument from the proof of the Douglas-Radò theorem (cf. [2, Chpt. 4.3]) involving the Courant-Lebesgue Lemma and the monotonicity of the boundary maps $X^R : \partial H \cap D_r(0) \to \Gamma, R > e^{r/c}$, it follows on the basis of the local estimates in Corollary 2.5 that the boundary values $X^R|\partial H \cap D_r(0)$ are equibounded and equicontinuous. Since $r > 0$ is arbitrary we may then apply standard compactness theorems for harmonic functions to obtain the existence of a limit map $X \in C^0(\bar{H}, \mathbb{R}^3) \cap C^\infty(H, \mathbb{R}^3)$ which is harmonic, maps ∂H into Γ weakly monotonically, and satisfies the conformality relations $|X_u|^2 = |X_v|^2,\ \langle X_u, X_v\rangle = 0$. Then, by another well know argument from the proof of the Douglas-Radò theorem, it follows that X cannot be constant on any interval of ∂H unless X is constant globally. But since $X([-1,1]) = \Gamma_1,\ X$ is not constant and hence the boundary map $X : \partial H \to \Gamma$ is strictly monotonic.

(ii) As shown in (i), X is at least a *branched* immersion, implying that the points in H where X fails to be immersed are isolated. Then, in view of the stability of the minimal immersion X^R the estimates of Schoen [14] for the induced metrics of the surfaces X^R become applicable and show that the limit X must again be immersed on H. □

It remains to prove the properness of the limit maps X.

Lemma 2.7 *Let* X *be a limit surface as in Proposition 2.6. Then one has the estimate*

$$E(X,\, \Omega(X,\rho)) \leq \left(\frac{1}{2}\alpha + \int\limits_{\Gamma} \frac{\sqrt{1 - \langle \gamma(s),\, \Gamma'(s)\rangle^2}}{\max(\rho, |\Gamma(s)|)}\, ds \right) \rho^2$$

with $\alpha := \sphericalangle(\gamma^-, \gamma^+)$.

Proof. Let $X = \lim_{k\to\infty} X^{R_k}$, let C be any compact subset of $\Omega(X,\rho)$ and let $\varepsilon > 0$. It follows from the local uniform convergence of (X^{R_k}) that $C \subset \Omega(X^{R_k}, \rho+\varepsilon)$ for sufficiently large k and hence, by Lemma 2.2

$$E(X^{R_k}, C) \leq \left(\frac{1}{2}\alpha(R_k) + \varepsilon(R_k) + \int\limits_{\Gamma\cap B_{R_k}} \frac{\sqrt{1-\langle\gamma(s),\,\Gamma'(s)\rangle^2}}{\max(\rho+\varepsilon, |\Gamma(s)|)}\, ds \right) (\rho+\varepsilon)^2 .$$

It follows from Lemma 2.1 (v) that $\alpha(R_k) \to \alpha (k \to \infty)$ and, as shown after (2.5) above, $\varepsilon(R_k) \to 0$ $(k \to \infty)$, which proves the lemma. $\square$

Lemma 2.8 *Every component of* $\Omega(X,\rho)$, $\rho \geq 1$, *is bounded.*

Proof. Let us first show that X is unbounded, i.e. $\Omega(X,\rho) \neq \bar{H}$ for any ρ. Assuming the contrary, it follows from Lemma 2.6 that $E(X,H) < +\infty$ and we may apply the lemma of Courant-Lebesgue to obtain

$$(2.9) \qquad (|X(r) - X(-r)|^2 \leq \left(\int_{-\pi}^{+\pi} \left| \frac{\partial X}{\partial\theta}(r\,e^{i\theta}) \right| d\theta \right)^2 \leq \frac{2\pi}{\ln\, r}\, E(X,H)$$

for arbitrary large radii r. Since $X([-1,1]) = \Gamma_1$ and $X : \partial H \to \Gamma$ is strictly monotonic, the points $X(r)$ and $X(-r)$ are contained in different components of $\Gamma\backslash\Gamma_1$ if $r > 1$. Therefore (2.8) contradicts the hypothesis (1.1) if r is large enough.

Let us now assume that Ω_0 is an unbounded component of $\Omega(X,\rho)$ for some $\rho \geq 1$, let $\rho' > \rho$ and let Ω' be the component of $\Omega(X,\rho')$ which contains Ω_0. Then Ω' is unbounded, too. For any sufficiently large $r > 1$ the lemma of Courant-Lebesgue provides $s \in (r, r^2)$ such that

$$(2.10) \qquad \mathrm{length}\, X(\Omega' \cap \partial D_s) \leq \left(\frac{2\pi}{\ln\, r}\, E(X,\Omega') \right)^{1/2} < \frac{1}{2}(\,\rho' - \rho)\,.$$

If, for such s, we had $H \cap \partial D \subset \Omega'$, then we would, for sufficiently large r, obtain the same contradiction to (1.1) as in the first part of the proof. Since Ω' is unbounded and connected, $\Omega' \cap \partial D_s$ cannot be empty provided that

s is large enough and we may therefore conclude that $H \cap \partial D_s \cap \partial\Omega' \neq \emptyset$ for s from (2.10). Since $|X| = \rho'$ on $\partial\Omega' \cap H$ it follows then from (2.10) that $|X| > \rho$ on $\Omega' \cap \partial D_s$. For large enough r we may clearly assume that $\Omega_0 \cap D_s \neq \emptyset$ and it follows then that $\Omega_0 \subset \Omega' \cap D_s$ since otherwise ∂D_s would separate Ω_0 within Ω'. We thus showed that Ω_0 is bounded, a contradiction. □

Lemma 2.9 *X is proper.*

Proof. We must show that $\Omega(X,\rho)$ is bounded for all large ρ. We consider $\rho \geq \rho_0$, where ρ_0 is given by Lemma 2.1. Let us recall that $C(X,\rho)$ is the connected component of $\Omega(X,\rho)$ which contains $[-1,1]$ and that $\Gamma_\rho = \Gamma \cap B_\rho$ for $\rho \geq \rho_0$. Therefore, $X : \partial H \to \Gamma$ being monotonic, it follows that $\Omega(X,\rho) \cap \partial H \subset C(X,\rho)$ for $\rho \geq \rho_0$. Thus there is only one component of $\Omega(X,\rho)$ which has nonempty intersection with ∂H, namely $C(X,\rho)$. Let us now consider a component Ω_0 of $\Omega(X,\rho)$ which is different from $C(X,\rho)$ and wich contains some point w with $|X(w)| \leq \rho/2$. Then $X(\Omega_0) \cap B_{\rho/2}(X(w))$ is a minimal surface which passes through $X(w)$ and has no boundary in the ball $B_{\rho/2}(X(w))$ and hence

$$A(X(\Omega_0)) \geq \pi(\rho/2)^2\,,$$

by the monotonicity formula. In view of Lemma 2.7 only finitely many such components Ω_0 can therefore exist, say $\Omega_1, \dots, \Omega_\ell$. On the complement of $C(X,\rho) \cup \Omega_1 \cup \cdots \cup \Omega_\ell$ we have the inequality $|X| > \rho/2$ and hence $\Omega(X,\rho/2) \subset C(X,\rho) \cup \Omega_1 \cup \cdots \cup \Omega_\ell$ and thus $\Omega(X,\rho/2)$ is bounded as follows from Lemma 2.8. □

Proof of Theorem 1.1. The theorem is a direct consequence of Proposition 2.6 and Lemma 2.9. As to Remark 1.1, the results of Meeks-Yau [10] apply directly to the surfaces X^R, showing that they are embeddings. The limit surface X is then embedded, too [6]. If Γ has finite total curvature, then the total curvature of the approximating curves $\Gamma_R \cup \beta_R$ remains uniformly bounded implying that the total curvature of the surfaces X^R remains uniformly bounded, by virtue of the Gauss-Bonnet theorem. Passage to the limit proves Remark 1.4. □

3 The asymptotical behavior

In this last section we describe the behavior at infinity of the surfaces constructed in section 2 by investigating their blow down limit $\lim_{\rho\to+\infty} \frac{1}{\rho}X(H)$. The corresponding boundary curves are the curves Γ^ρ, given in arc-length parametrization by

$$\Gamma^\rho(s) = \frac{1}{\rho}\Gamma(\rho s)\,.$$

In order to obtain convergence of the surfaces $\frac{1}{\rho}X(H)$ in the sense of mappings, we must reparametrize these surfaces appropriately what we do by choosing a conformal automorphism φ^ρ of the upper half plane H such that

$$\varphi^\rho(0) = 0,\ \varphi^\rho(-1) = s^-(\rho),\ \varphi^\rho(1) = s^+(\rho)$$

where $\Gamma(s^\pm(\rho))$ are the endpoints of $\Gamma \cap B_\rho$. Then we define

$$Y^\rho := \frac{1}{\rho}X \circ \varphi^\rho,\ \rho \geq 1\,,$$

so that $Y^\rho(0) = 0$ and $Y^\rho(\pm 1)$ are the endpoints of $\Gamma^\rho \cap B_1$.

Let us at first compute the limit of the curves Γ^ρ, keeping in mind the statements in Lemma 2.1.

Lemma 3.1 $\lim_{\rho\to+\infty} \Gamma^\rho = \Gamma^\infty$ *uniformly on any compact arc-length interval where*

$$\Gamma^\infty(s) := \begin{cases} s\gamma^-, & -\infty < s \leq 0 \\ s\gamma^+, & 0 \leq s < +\infty\,. \end{cases}$$

Proof. For $s > 0$ we obtain

$$\Gamma^\rho(s) - s\gamma^+ = s\left(\frac{1}{\rho s}\Gamma(\rho s) - \gamma^+\right) = s\left(\frac{|\Gamma(\rho s)|}{\rho s}\gamma(\rho s) - \gamma^+\right) \to 0\,(\rho \to +\infty)$$

uniformly on any interval of the form $[\delta, \delta^{-1}],\ 0 < \delta < 1$, as follows from Lemma 2.1 (i) and (v). The corresponding statement holds for negative s. On the other hand, the Γ^ρ being uniformly Lipschitz and $\Gamma^\rho(0) = 0$, any sequence $\Gamma^{\rho_k},\ k \in \mathbb{N}$, contains a subsequence which converges uniformly on any compact interval. This proves the lemma. $\square$

As for the convergence of Y^ρ, the cases $\gamma^+ = \gamma^-$ and $\gamma^+ \neq \gamma^-$ clearly have to be distinguished. We start with the first case and may assume that $\gamma^+ = \gamma^- = (1,0,0)$. Let C_ε be a circular cone with axis $\{(t,0,0)|\, t \geq 0\}$, vertex at the origin, and opening angle $\varepsilon > 0$. Since $\gamma^+ = \gamma^- \in C_\varepsilon$ we may choose $R = R(\varepsilon) > 0$ such that $\Gamma \subset (-R,0,0) + C_\varepsilon$. By the convex hull property of minimal surfaces the approximating compact surfaces X^R of section 2 and hence the limit surfaces X must be contained in $(-R,0,0)+C_\varepsilon$. It follows that

$$Y^\rho(H) \subset (-R/\rho, 0, 0) + C_\varepsilon \,.$$

Since $\varepsilon > 0$ is arbitrary, we see that the surfaces Y^ρ converge to the half line $\{(t,0,0)|\, t \geq 0\}$ and there is no limit surface. It remains to consider the case that $\gamma^+ \neq \gamma^-$. Using the notation of section 2 we have

$$\Omega(Y^\rho, R) = (\varphi^\rho)^{-1}(\Omega(X, \rho R))$$

and we therefore obtain from Lemma 2.7 and the conformal invariance of Dirichlet's energy the estimate

$$E(Y^\rho,\ \Omega(Y^\rho,\ R) \leq \left(\frac{1}{2}\alpha + \int\limits_\Gamma \frac{\sqrt{1 - \langle \gamma(s),\, \Gamma'(s)\rangle^2}}{\max(\rho R,\, |\Gamma(s)|} \, ds \right) R^2 \,, \tag{3.1}$$

so that we have an energy bound uniformly for $\rho \geq 1$; actually, the integral over Γ in (3.1) tends to 0 if $\rho \to +\infty$. Since moreover the hypotheses (1.2) for the boundary curve Γ hold in a uniform way for the curves $\Gamma^\rho,\ \rho \geq 1$ (for (1.2) (i) the assumption that $\gamma^+ \neq \gamma^-$ enters), the analysis of section 2 applies and we may conclude that any sequence $Y^{\rho_k},\ k \in \mathbb{N}$, where $\rho_k \to +\infty$ contains a subsequence which converges uniformly on any compact subdomain of $\bar{H}$ and locally smoothly in the interior of H to a proper limit map $Y \in C^0(\bar{H},\, \mathbb{R}^3) \cap C^\infty(H, \mathbb{R}^3), Y$ is conformal minimal immersion, and $Y|\partial H$ parametrizes Γ^∞. Moreover, passing with $\rho \to \infty$ in (3.1) we see that any such Y satisfies the estimate

$$E(Y,\, \Omega(Y, R)) \leq \frac{1}{2}\alpha\, R^2 \,. \tag{3.2}$$

We would like to argue that $Y(H)$ is a planar sector of opening angle α. In case $\alpha = \pi$ the boundary curve Γ^∞ is a full straight line and we can extend

Y by reflection across Γ^∞ to obtain a properly immersed minimal surface M which by (3.2) satisfies

$$A(M \cap B_R) \le \pi R^2,\ R > 0\,. \tag{3.3}$$

It follows from the monotonicity formula that we have equality in (3.3), from what it is not difficult to see that M must be a plane. We are thus left with the case that γ^- and γ^+ are linearly independent. We may clearly assume that

$$\gamma^- = (\cos\alpha/2,\ -\sin\alpha/2,\ 0),\ \gamma^+ = (\cos\alpha/2,\ \sin\alpha/2,\ 0)$$

where, as before, $\alpha = \sphericalangle(\gamma^-, \gamma^+) \in (0, \pi)$. For $\varepsilon \in (0, \pi - \alpha)$ let us define the cone

$$C_\varepsilon = \{(x_1, x_2, x_3)|\ x_1 \ge 0, |x_2| \le \left(\tan\frac{\alpha+\varepsilon}{2}\right) x_1,\ |x_3| \le \left(\tan\frac{\varepsilon}{2}\right) x_1\}\,.$$

Since the ends of the curve Γ are contained in C_ε we may choose $R > 0$ depending on ε so that $\Gamma \subset (-R, 0, 0) + C_\varepsilon$. From the convexity of C_ε we may conclude as before that $X(H) \subset (-R, 0, 0) + C_\varepsilon$ and hence $Y^\rho(H) \subset (-R/\rho, 0, 0) + C_\varepsilon$. It follows that any limit map Y satisfies $Y(H) \subset C_\varepsilon$. Since $\varepsilon > 0$ is arbitrary small, we infer that $Y(H)$ is a planar sector of opening angle α. We thus have shown

Theorem 3.2 *Let X be a minimal surface with boundary Γ as constructed in section 2 and let γ^- and γ^+ be as defined in Lemma 2.1. If $\gamma^- \ne \gamma^+$ the surfaces $\frac{1}{\rho}X$ tend to a limit surface Y as $\rho \to 0$ which is a convex planar sector between the half lines determined by γ^- and γ^+. The convergence is in the sense of mappings if suitable conformal parametrizations are chosen for the family $\frac{1}{\rho}X$, $\rho > 0$. In case $\gamma^- = \gamma^+$ the surfaces $\frac{1}{\rho}X$ shrink to the half line determined by $\gamma^- = \gamma^+$.*

Bibliography

[1] Alt, H.W., Verzweigungspunkte von H-Flächen. I. Math. Z. 127(1972), 333-362, II. Math. Ann. 201 (1973), 33-55

[2] Dierkes, U., Hildebrandt, S. Küster, S., Wohlrab O., Minimal Surfaces I, II. Springer Verlag Berlin Heidelber New York, 1992

[3] Ekholm, T., White, B., Wienholtz, D., Embeddedness of minimal surfaces with total boundary curvature at most 4π. Ann. Math. 155(2002), 209-234

[4] Ferrer, L., Martín, F., Properly embedded minimal disks bounded by noncompact polygonal lines. Pacific J. Math. 214 (2004), 55-88

[5] Gulliver, R., Regularity of minimizing surfaces of prescribed mean curvature. Ann. Math. 97 (1973), 275-305

[6] Gulliver, R., Spruck, J., On embedded minimal surfaces. Ann. Math. 103 (1976) 331-347, with a correction in Ann. Math. 109 (1979), 407-412

[7] Jenkins, H., Serrin, J., Variational problems of minimal surface type. III. The Dirichlet problem with infinite data. Arch. Ration. Mech. Anal. 29 (1968), 304-322

[8] López, F.J., Wei, F., Properly immersed minimal disks bounded by straight lines. Math. Ann. 318 (2000), 667-706

[9] López, F.J., Martín, F., A Uniqueness theorem for properly embedded minimal surfaces bounded by straight lines. J. Austr. Math. Soc. (Series A) 69 (2000), 362-402

[10] Meeks, W.H., Yau, S.-T., Topology of three-manifolds and the embedding problems in minimal surface theory. Ann. Math. 112 (1980), 441-484

[11] Neovius, E. R., Analytische Bestimmung einiger ins Unendliche reichender Minimalflächenstücke, deren Begrenzung gebildet wird von drei geraden Linien, von denen zwei sich schneiden, während die dritte der Ebene der beiden ersten parallel ist. Schwarz-Festschrift. Springer, Berlin 1914

[12] Osserman, R., A proof of the regularity everywhere of the classical solution to Plateau's problem. Ann. Math. 91 (1970), 550-569

[13] Riemann, B., Über die Fläche vom kleinsten Inhalt bei gegebener Begrenzung. Bernhard Riemann's gesammelte Mathematische Werke und wissenschaftlicher Nachlass. B. G. Teubner, Leipzig, 1876, p. 283-315

[14] Schoen, R., Estimates for stable minimal surfaces in three dimensional manifolds, Seminar on Minimal Submanifolds, edited by Enrico Bombieri, Ann. Math. Stud. 103 (1983), 111-126

[15] Simon, L., Lectures on Geometric Measure Theory. Proceedings of the Centre for Mathematical Analysis, Australian National University, Vol. 3, 1983

[16] Tomi, F., Ye, R., The exterior Plateau problem. Math. Z. 205 (1990), 233-245